1 Forced Damped Harmonic Motion

Suppose that we add forcing to a forced damped harmonic oscillator,

\[ x'' + 2cx' + \omega_0^2 x = A \cos \omega t. \]  \hspace{1cm} (1)

The associated homogeneous equation is

\[ x'' + 2cx' + \omega_0^2 x = 0. \]  \hspace{1cm} (2)

The characteristic polynomial of (2),

\[ P(\lambda) = \lambda^2 + 2c\lambda + \omega_0^2, \]  \hspace{1cm} (3)

has roots

\[ \lambda = -c \pm \sqrt{c^2 - \omega_0^2}. \]

In the underdamped case \( c < \omega_0 \), the solution to the homogeneous equation (2) is

\[ x_h = e^{-ct} (c_1 \cos \eta t + c_2 \sin \eta t), \]

where \( \eta = \sqrt{\omega_0^2 - c^2} \).

To look for a particular solution to (1), we use the complex method. That is, we shall look for a solution of the form \( x_c = Ae^{i\omega t} \) to the equation

\[ x_c'' + 2cx_c' + \omega_0^2 x_c = A e^{i\omega t}, \]  \hspace{1cm} (4)
and then set $x_p$ equal to the real part of our solution. Substituting $x_c$ into the right-hand side of (4), we obtain

$$x_c'' + 2cx_c' + \omega_0^2 x_c = [(i\omega)^2 + 2c(i\omega) + \omega_0^2]ae^{i\omega t} = P(i\omega)ae^{i\omega t},$$

where $P$ is the characteristic polynomial (3). Thus, equation (4) becomes

$$P(i\omega)ae^{i\omega t} = Ae^{i\omega t},$$

and

$$x_c(t) = ae^{i\omega t} = \frac{A}{P(i\omega)}e^{i\omega t} = H(i\omega)Ae^{i\omega t},$$

where

$$H(i\omega) = \frac{1}{P(i\omega)}.$$

2 The Transfer Function

We say that $H(\lambda) = 1/P(\lambda)$ is the transfer function. Let us examine $H(i\omega)$ more closely. First,

$$P(i\omega) = (i\omega)^2 + 2c(i\omega) + \omega_0^2 = (\omega_0^2 - \omega^2) + 2ic\omega.$$

If we write $P(i\omega)$ in polar form, then

$$P(i\omega) = Re^{i\phi} = R(\cos \phi + i \sin \phi),$$

where

$$R = \sqrt{(\omega_0^2 - \omega^2)^2 + 4c^2\omega^2}$$

and $\phi$ is the angle defined by the equations

$$\cos \phi = \frac{\omega_0^2 - \omega^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4c^2\omega^2}},$$

$$\sin \phi = \frac{2c\omega}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4c^2\omega^2}}.$$

Since $2c\omega > 0$, we know that $\sin \phi > 0$. Equivalently, $0 < \phi < \pi$. Thus,

$$\phi = \phi(\omega) = \cot^{-1}\left(\frac{\omega_0^2 - \omega^2}{2c\omega}\right).$$

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Therefore, we can write the transfer function as
\[ H(i\omega) = \frac{1}{P(i\omega)} = \frac{1}{R} e^{-i\phi}. \]
We define the gain to be
\[ G(\omega) = \frac{1}{R} = \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4c^2\omega^2}}, \]
and we will rewrite the transfer function as
\[ H(i\omega) = G(\omega)e^{-i\phi(\omega)}. \]
Thus, the solution to
\[ x'' + 2c x' + \omega_0^2 x = Ae^{i\omega t} \]
is
\[ x_c(t) = H(i\omega)Ae^{i\omega t} = G(\omega)Ae^{i(\omega t - \phi)} \]
and our particular solution is
\[ x_p(t) = \text{Re}(x_c(t)) = G(\omega)A \cos(\omega t - \phi). \]
It is now clear that \( x_p \) is sinusoidal with the same frequency as the forcing term. In addition, \( x_p \) is out of phase with the driving force by the amount
\[ \phi = \phi(\omega) = \cot^{-1}\left(\frac{\omega_0^2 - \omega^2}{2c\omega}\right). \]

3 The General Solution

The general solution to
\[ x'' + 2c x' + \omega_0^2 x = A \cos \omega t \]
is
\[ x(t) = x_h(t) + x_p(t) = e^{-ct}(c_1 \cos(\eta t) + c_2 \sin(\eta t)) + G(\omega)A \cos(\omega t - \phi). \]
Since \( x_h \) has the factor \( e^{-ct} \), the homogeneous part of the solution quickly decays to zero as \( t \to \infty \). For this reason, \( x_h \) is called the transient term while \( x_p \) is called the steady-state term.
4 Amplitude and Phase

Let us examine the amplitude and phase of the steady-state solution,

\[ x_p(t) = G(\omega)A \cos(\omega t - \phi), \]

where

\[ G(\omega) = \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4c^2\omega^2}}. \]

Now let \( \omega = s\omega_0 \) and \( D = 2c/\omega_0 \). These new constants, \( s \) and \( D \), measure the ratio of the driving frequency to the natural frequency and the effect of the damping force, respectively. Thus,

\[ G = \frac{1}{\omega_0^2 \sqrt{(1 - s^2)^2 + D^2 s^2}} \]

or

\[ \omega_0^2 G = \frac{1}{\sqrt{(1 - s^2)^2 + D^2 s^2}}. \]

This expression shows us how the gain varies as \( s = \omega/\omega_0 \) varies. The natural frequency is fixed in

\[ x'' + 2cx' + \omega_0^2 x = A \cos \omega t \]

and \( D = 2c/\omega_0 \) is proportional to the damping constant.